

D-branes in nongeometric backgrounds

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ABSTRACT: “T-fold” backgrounds are generically-nongeometric compactifications of string theory, described by T^n fibrations over a base N with transition functions in the perturbative T-duality group. We review Hull’s doubled torus formalism, which geometrizes these backgrounds, and use the formalism to constrain the D-brane spectrum (to leading order in g_s and α') on T^n fibrations over S^1 with $O(n, n; \mathbb{Z})$ monodromy. We also discuss the (approximate) moduli space of such branes and argue that it is always geometric. For a D-brane located at a point on the base N , the classical “D-geometry” is a T^n fibration over a multiple cover of N .

KEYWORDS: D-branes, Superstring Vacua.

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1. Introduction

In $\mathcal{N} = 1$, $d = 4$ compactifications of type IIB with flux, superpotentials for the complex structure moduli are induced by turning on various combinations of 3-form flux [1–8]. In type IIA string theory [9, 10], all geometric moduli can be stabilized by NS-NS and RR fluxes. In particular, NS-NS 3-form flux together with nonvanishing torsion generates a superpotential for the complex structure at tree level [11, 12].¹ In addition, “geometric” flux can generate terms in the superpotential coupling Kähler and complex structure moduli [16–20].

In the absence of IIA NS-NS flux, the IIA complex structure moduli and IIB Kähler moduli are exchanged by mirror symmetry. Thus the mirror of NS-NS flux, when it exists,

¹This also holds for the heterotic string, since it involves the common NS-sector. [13–15].

provides a natural mechanism in type IIB for stabilizing the Kähler moduli [11, 20]. The resulting compactifications, and other non-Calabi-Yau geometries, have been the subject of much recent interest [21–28].

For Calabi-Yau manifolds with special Lagrangian T^3 fibrations, mirror symmetry corresponds to T-duality along the T^3 [29]. T-duality maps NS-NS 3-form flux H to geometric “twisted torus” backgrounds, or to nongeometric backgrounds [20, 30–33]. If H has one index polarized along this T^3 , the mirror manifold [34, 35] is a “half-flat” $SU(3)$ -structure manifold [36–42]. If H has two or three indices polarized along this T^3 , the resulting T-dual is not geometric. In the particular case that two indices of H are polarized along the T^3 , the T-dual is known to be a T^2 fibration over S^1 with a nongeometric monodromy in the T-duality group $O(2, 2; \mathbb{Z})$. This corresponds to a class of nongeometric flux studied in [30, 20, 43–49]. Such nongeometric string theory backgrounds are also interesting in their own right and remain relatively unexplored.²

D-branes in such backgrounds are central to the physics of these models. They can provide nontrivial gauge dynamics; space-filling branes can lead to open string gauge fields, or wrapped branes can behave as W-bosons. In addition, D-branes provide an alternate definition of the background geometry in terms of the moduli space of the probe. Wrapped D-brane states are important clues to the nonperturbative structure of the theory; they constitute part of the nonperturbative spectrum important for studying string duality, and they indicate what vacua are connected in field space [11, 51]. Finally, D-branes are crucial in our understanding of mirror symmetry [29, 52].

In this paper we study D-branes in “T-fold” (also known as “monodrofold”) compactifications, which are a generalization of the nongeometric compactifications discussed above. T-folds are T^n fibrations over a base N with transition functions in the perturbative T-duality group G_T . While T-folds are locally describable as geometric manifolds on N , there is not a globally geometric description. Such manifolds have been studied by a number of authors [50, 53–58]. For bosonic strings, Hull [54] has proposed a geometrization by doubling the T^n to $T^{2n} \supset T^n$, which linearizes the $G_T = O(n, n; \mathbb{Z})$ action. The D-branes on T^n are specified by n -dimensional submanifolds of T^{2n} .³ While the worldsheet quantum mechanics of [54] is not well-understood, we already find interesting restrictions on the D-brane spectrum at the classical level, i.e., to lowest order in α' and g_s .

We will focus on T-fold fibrations over S^1 . The classically allowed, locally geometric D-branes for these models are straightforward to classify. When the brane wraps the base S^1 , the monodromy projects out large classes of D-branes. When the brane is at a point on the S^1 , there is no such projection. In the latter case, we find that the classical moduli space of the D-branes is always a geometric T^n fibration — the Dirichlet T^n bundle — over a multiple or infinite cover of the S^1 . Similarly, for branes (multiply) wrapping the S^1 , the classical moduli space is the space of Wilson lines on the Neumann T^n bundle over (a multiple cover of) the S^1 and is again geometric. These statements generalize in the natural way for D-branes in an arbitrary T-fold.

²In the case where H has three indices polarized along the T^3 , the T-dual is not even locally geometric [20, 50].

³This description of D-branes is much like that for generalized Calabi-Yau backgrounds [59–61].

The plan of this paper is as follows. In section 2 we review Hull’s formalism, developed in [54], for describing twisted torus compactifications, with particular attention paid to torus fibrations over a circle. This review is a partial reworking of Hull’s discussion and makes explicit some points that were implicit in [54]. In section 3 we review D-branes in these backgrounds, describe the allowed topological classes of geometric D-branes for torus fibrations over a circle, and discuss the moduli space of D0-branes. We then discuss two examples in particular: T^3 with nonvanishing NS-NS 3-form flux, together with the T-duals described by [20, 30, 58]; and asymmetric orbifolds describable as T-folds [53, 55, 56]. In section 4 we conclude by presenting some possible directions for further study.

2. Review of Hull’s formalism

Hull [54] has studied a class of perturbative string compactifications which can be described as T^n fibrations over a base N . To construct these fibrations, we begin by compactifying a D -dimensional string theory down to $D - n$ dimensions on T^n . The T-duality symmetries of these compactifications are the basis of T-fold constructions. In bosonic string theory, with $D = 26$, this compactification has a perturbative T-duality group $G_T = O(n, n; \mathbb{Z})$. The set of large diffeomorphisms $GL(n, \mathbb{Z})$ of T^n is a subgroup of $O(n, n; \mathbb{Z})$. For type II strings ($D = 10$), some elements of the $O(n, n; \mathbb{Z})$ group exchange type IIA and IIB theories. These are not symmetries of the worldsheet theory; in these cases we will take G_T to be the subgroup $SO(n, n; \mathbb{Z})$ which maps IIA(B) to IIA(B).⁴

In the models we study, we compactify further on a base manifold N . The moduli of the $(D - n)$ -dimensional string compactification are permitted to vary over N , so that the full string background satisfies the worldsheet beta function equations. One starts by covering N with a set of coordinate patches U_α . For any two intersecting patches U_α, U_β the background fields on T^n must be identified on $U_{\alpha\beta} = U_\alpha \cap U_\beta$, by an action of the structure group. When the structure group is the semidirect product of $GL(n, \mathbb{Z})$ and $U(1)^n$ (the latter being the translations in the T^n directions), the resulting fibration is geometric. T-folds correspond to fibrations with structure group G_T . For the remainder of this paper, we will take G_T to be $O(n, n; \mathbb{Z})$ or $SO(n, n; \mathbb{Z})$.

Let us first describe the essential point of [54], for the bosonic string theory. Hull geometrizes the action of $O(n, n; \mathbb{Z})$ by introducing a “doubled torus” T^{2n} . There is a signature (n, n) metric L on T^{2n} , and the “physical” T^n , describing the target space of the string, is a null submanifold with respect to L . $O(n, n; \mathbb{Z})$ is the set of linear transformations on T^{2n} which preserves both the periodicities of the coordinates and the metric L . The T-fold can be described as a geometric fibration of T^{2n} over N , with transition functions in $O(n, n; \mathbb{Z}) \subset GL(2n; \mathbb{Z})$. The string compactification is described by a sigma model on

⁴This group can be identified by studying the generators of $O(n, n; \mathbb{Z})$ as in section 2.4 of [62]. These generators are: large diffeomorphisms of the torus, which have determinant 1 as elements of $O(n, n; \mathbb{Z})$; shifts of the B-field which have determinant 1; and T-dualities along any leg which have determinant -1 . The elements of $O(n, n; \mathbb{Z})$ which take type IIA(B) back to type IIA(B) must be made up of group elements which have an even number of T-dualities when written in terms of the generators. These are all the group elements with determinant 1.

this $(26 + n)$ -dimensional space, combined with a self-duality constraint which projects out n of the coordinates to leave us with a critical string theory.

At present the quantization of the worldsheet theory is only partially understood at best. Furthermore, the discussion here and in [54] does not include worldsheet fermions. These issues should be addressed. In this paper, however, we confine our attention to the classical worldsheet theory, for which specifying the equations of motion and boundary conditions suffice; furthermore, we restrict our study to the bosonic degrees of freedom.

2.1 Basic description of the doubled torus

2.1.1 Fields and equations of motion

The essence of [54] is to present the worldsheet equations of motion and boundary conditions in a manifestly $O(n, n; \mathbb{Z})$ -symmetric manner, building on the earlier work of refs. [63–65]. One begins with a Lagrangian for $2n$ fields, and derives equations of motion and boundary conditions for the fields. Next, one imposes a self-duality constraint on the solutions to the equations of motion, which cuts the degrees of freedom in half. This classical analysis will suffice at lowest order in α' and g_s .

We wish to describe a T^n fibration over an $(d - n)$ -dimensional base N . We begin with a set of two-dimensional fields $\mathbb{X}^I(\sigma^\alpha), Y^A(\sigma^\alpha)$, with (σ^0, σ^1) the worldsheet coordinates, $I = 1, \dots, 2n$, and $A = 1, \dots, d - n$. The classical Lagrangian for the bosonic coordinates on the flat worldsheet with metric $\eta = \text{diag}(-1, 1)$ can be written as

$$\mathcal{L} = -\frac{1}{2}\mathcal{H}_{IJ}(Y)\eta^{\alpha\beta}\partial_\alpha\mathbb{X}^I\partial_\beta\mathbb{X}^J - \eta^{\alpha\beta}\mathcal{J}_{IA}(Y)\partial_\alpha\mathbb{X}^I\partial_\beta Y^A + \mathcal{L}_N(Y), \quad (2.1)$$

with \mathcal{H} a positive definite metric and $\mathbb{X} \in \mathbb{R}^{2n}/\Gamma \equiv T^{2n}$, where $\Gamma \subset \mathbb{R}^{2n}$ is a $2n$ -dimensional lattice. The quantity $\mathcal{J}_{IA}dY^A$ determines the connection for the T^{2n} fibration over N as discussed in [54]. We assume a flat worldsheet and therefore ignore the dilaton coupling. Nevertheless, we will have to specify the dilaton since it transforms nontrivially under T-duality, as we describe below.

Locally on N , given a choice of physical subspace $T^n \subset T^{2n}$, the T^{2n} metric \mathcal{H} can be written in terms of the physical T^n metric G and the NS-NS 2-form potential B . Thus, G and B are completely geometrized to \mathcal{H} in the doubled formalism. There is no antisymmetric tensor term of the form $\mathcal{B}_{IJ}\partial_0\mathbb{X}^I\partial_1\mathbb{X}^J$ in the Lagrangian (2.1), i.e., no $2n$ -dimensional \mathcal{B} -field analogous to the B -field in the standard sigma model. Such couplings will arise after an additional self-duality constraint is solved (cf. eq. (2.3) below).

The equations of motion that follow from varying (2.1) are

$$\eta^{\alpha\beta}\partial_\alpha(\mathcal{H}_{IJ}\partial_\beta\mathbb{X}^J + \mathcal{J}_{IA}\partial_\beta Y^A) = 0. \quad (2.2)$$

The Lagrangian and equations of motion are $\text{GL}(2n; \mathbb{Z})$ invariant. However, one also imposes the self-duality constraint

$$\partial_\alpha\mathbb{X}^I = L^{IJ}\epsilon_\alpha{}^\beta(\mathcal{H}_{JK}\partial_\beta\mathbb{X}^K + \mathcal{J}_{JA}\partial_\beta Y^A), \quad (2.3)$$

which breaks the invariance down to $O(n, n; \mathbb{Z})$. Here $\epsilon_{\alpha\beta}$ is the two-dimensional antisymmetric tensor. L^{IJ} is a constant, invertible, symmetric matrix, invariant under

$O(n, n; \mathbb{Z}) \subset GL(2n; \mathbb{Z})$. Eq. (2.3) cuts the number of on-shell degrees of freedom in half. The matrix LH has n positive and n negative eigenvalues; so, physically, the constraint requires that half of the $2n$ coordinates be left-moving and half right-moving.

We impose the constraint (2.3) after first varying (2.1) to obtain the equations of motion.⁵ The curl of (2.3) gives back the equations of motion (2.2), and therefore vanishes. In the presence of boundaries, the allowed boundary conditions follow from (2.1) as well; we will discuss them in section 3.

A nontrivial fibration over N means that nontrivial elements of $O(n, n; \mathbb{Z})$ relate \mathcal{H} , \mathcal{J} and \mathbb{X} over different coordinate patches of N . Therefore, we must specify the action of the structure group $O(n, n; \mathbb{Z})$ on these quantities. The matrix $g^I{}_J$ acts on lower indices from the right, while the inverse matrix $(g^{-1})^I{}_J$ acts on upper indices from the left. Define L_{IJ} to be the inverse of L^{IJ} , such that $L_{IJ}L^{JK} = \delta_I^K$. A transformation in $O(n, n)$ is defined as one which preserves L_{IJ} :

$$(g^t)_I{}^K L_{KM} g^M{}_J \equiv g^t L g = L_{IJ}. \tag{2.4}$$

Then \mathcal{H} , \mathbb{X} , \mathcal{J} transform as

$$\begin{aligned} \mathcal{H}_{IJ} &\rightarrow \mathcal{H}'_{IJ} = (g^t \mathcal{H} g)_{IJ}, \\ \mathbb{X}^I &\rightarrow \mathbb{X}'^I = (g^{-1})^I{}_J \mathbb{X}^J, \\ \mathcal{J}_{IA} &\rightarrow \mathcal{J}'_{IA} = (g^t)_I{}^K \mathcal{J}_{KA}. \end{aligned} \tag{2.5}$$

We require that g preserve the lattice Γ , which breaks $O(n, n)$ down to $O(n, n; \mathbb{Z})$.

2.1.2 Choosing a polarization

At fixed Y , we define a “physical subspace” of T^{2n} to be a $T^n \subset T^{2n}$ that is null with respect to the metric L . A choice of physical subspace is equivalent to a choice of $GL(n)$ in $O(n, n)$, or alternatively, a choice of coordinate basis for which L has the form

$$L = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{2.6}$$

where $\mathbf{1}$ is the $n \times n$ identity matrix. This change of basis can be implemented by the $(2n \times 2n)$ -dimensional matrix

$$\Pi = \begin{pmatrix} \Pi^i{}_I \\ \tilde{\Pi}_{iI} \end{pmatrix}, \tag{2.7}$$

where $i = 1 \dots n$. The lowercase indices are in different positions in these two projectors to indicate that we have decomposed a $2n$ -dimensional representation of $O(n, n)$ into an n -dimensional representation (the upper indices) of $GL(n)$ and a dual (inverse transpose)

⁵To impose eq. (2.3) as a constraint in the path integral, via a Lagrange multiplier, would imply that the configuration space is the space of classical solutions. One would like to have a clean Lagrangian formulation from which both the equations of motion and the constraints follow. At the expense of sacrificing manifest 2d Lorentz invariance, this could be sought in a slightly modified version of Tseytlin’s formalism [66, 67], in which the self-duality constraint (2.3) is an equation of motion.

n -dimensional representation (the lower indices), i.e., $2n \rightarrow n \oplus n'$. As this corresponds to a basis such that L has the form (2.6), the $\Pi, \tilde{\Pi}$ are themselves null by construction:

$$L^{IJ}\Pi^i_I\Pi^j_J = L^{IJ}\tilde{\Pi}_i_I\tilde{\Pi}_j_J = 0. \tag{2.8}$$

We can write X in terms of the ‘‘physical’’ coordinates $X^i = \Pi^i_I\mathbb{X}^I$ and the ‘‘dual’’ coordinates $\tilde{X}_i = \tilde{\Pi}_i_I\mathbb{X}^I$. In this coordinate system, the $O(n, n)$ invariant metric is

$$ds_L^2 = d\mathbb{X}^I d\mathbb{X}^J L_{IJ} = 2dX^i d\tilde{X}_j. \tag{2.9}$$

To make contact with the standard sigma model, an additional step is needed beyond the choice of polarization. We must restrict \mathcal{H} to be an $O(n, n)/(O(n) \times O(n))$ coset metric, that is, to $\mathcal{H} = V^t V$, where the vielbein V is a representative of this coset. Then, in this basis determined by the polarization, \mathcal{H} can be written as

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{ij} & \mathcal{H}_i{}^j \\ \mathcal{H}^i{}_j & \mathcal{H}^{ij} \end{pmatrix} = \begin{pmatrix} G_{ij} - B_{ik}G^{kl}B_{lj} & B_{ik}G^{kj} \\ -G^{ik}B_{kj} & G^{ij} \end{pmatrix}, \tag{2.10}$$

where G is the metric and B the NS-NS 2-form potential. After solving the self-duality constraints (2.3), one arrives at the standard sigma model.

2.1.3 Some $O(2, 2; \mathbb{Z})$ transformations

The utility of this formalism is that it makes the $O(n, n; \mathbb{Z})$ duality transformations relatively simple: the coordinates transform linearly in the $2n$ -dimensional defining representation of the group. To illustrate this, we now review a few basic examples of $O(2, 2; \mathbb{Z})$ transformations on T^2 , as also discussed in [54]. We will work in the basis described by (2.6)–(2.10).

- **T-duality on a single cycle.** T-duality exchanges a cycle of the torus T with a cycle of the dual torus \tilde{T} . These are generated by

$$g_{T,1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{T,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.11}$$

On complex tori, this exchanges the complexified Kähler structure $\rho = \int_{T^2}(B + iJ)$ and the complex structure τ . Note that $g_{T,1}$ and $g_{T,2}$ are not elements of the $SO(2, 2; \mathbb{Z})$ T-duality group of type IIA(B) string theory by themselves; that group consists of elements with even numbers of these generators.

- **Shift of B-field.** It is easy to check that the following transformation implements a shift of $\int_{T^2} B$ by N units, which is a symmetry of the closed string theory:

$$g_N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -N & 1 & 0 \\ N & 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

- **Geometric $GL(n, \mathbb{Z})$ transformations of T^n .** These are imposed by block-diagonal matrices of the form

$$g_{\text{geom}} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad (2.13)$$

where A is a 2×2 invertible matrix with integer entries. Note that these shifts and the B -field shifts generate a group of block-lower-triangular matrices (with the upper right 2×2 block set to zero).

Hull notes that one can describe T-duality via either an “active” or a “passive” transformation. In an active transformation the polarization $(\Pi, \tilde{\Pi})$ remains fixed; the metric data \mathcal{H}, \mathcal{J} transform as in (2.5), while the dilaton transforms as

$$e^\Phi \rightarrow e^{\Phi'} = \left(\frac{\det G'}{\det G} \right)^{1/4} e^\Phi. \quad (2.14)$$

In a passive transformation, the polarization and the dilaton are taken to transform (the latter just as in the active transformation), while the metric data stays fixed. These are physically equivalent; we will stick to describing T-duality via active transformations.

2.1.4 Building T-folds

The class of “T-fold” compactifications under study are locally T^n fibrations over patches of the base manifold N . Globally, the backgrounds can be described as follows. Let N be covered by a set of open, simply connected neighborhoods U_α . In each neighborhood U_α , the fibration is “geometric”: the polarization $(\Pi, \tilde{\Pi})$ is constant, while the metric \mathcal{H} can vary. If two sets U_α, U_β , with metrics $\mathcal{H}_\alpha, \mathcal{H}_\beta$ overlap in a set $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the data are related by an $O(n, n; \mathbb{Z})$ transformation $g_{\alpha\beta}$ acting on \mathcal{H}, \mathcal{J} as in (2.5):

$$\mathcal{H}_\beta = g_{\alpha\beta}^t \mathcal{H}_\alpha g_{\alpha\beta}, \quad \mathcal{J}_\beta = g_{\alpha\beta}^t \mathcal{J}_\alpha, \quad (2.15)$$

while the dilaton transformation is given by (2.14). These transformations must be consistent on triple overlaps $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \quad (2.16)$$

As we will see in an explicit example, nontrivial transformations g can lead to NS-NS flux, or the geometric and nongeometric fluxes discussed in [20, 30]. In these examples, we will call “geometric” fibrations those for which the transition functions g are generated by elements of $GL(n; \mathbb{Z})$ combined with integral shifts of the B -field. These two operations generate a proper subgroup of $O(n, n; \mathbb{Z})$ which does not contain any T-duality (inversion) transformations. A nongeometric fibration is simply one which is not geometric by this definition.

The simplest example of a torus fibration is a fibration over $N = S^1$. If the base coordinate is $Y = Y + 2\pi$, we simply demand that the monodromies live in $O(n, n; \mathbb{Z})$: $\mathcal{H}(Y + 2\pi) = g^t \mathcal{H}(Y) g$, $\mathcal{J}(Y + 2\pi) = g^t \mathcal{J}$. For a more general non-simply connected base N , we impose the same requirements for the monodromies associated to any noncontractible loop.

Once we have such a fibration, we can generate a set of equivalent string backgrounds by performing fiberwise T-duality on them: over each element of the base, we transform the metric and dilaton via (2.5), (2.14), by some $g_D \in \mathcal{O}(n, n; \mathbb{Z})$ which is constant along N . For this transformation to be consistent in the case of topologically nontrivial $O(n, n; \mathbb{Z})$ transformations, one will also have to conjugate the transition functions $g_{\alpha\beta}$ or the monodromy matrices $g_{2\pi}$ by g_D :

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = g_D^{-1} g_{\alpha\beta} g_D, g_{2\pi} \rightarrow g'_{2\pi} = g_D^{-1} g_{2\pi} g_D. \tag{2.17}$$

2.2 A note on S^1 fibrations over a one-dimensional base

The simplest example to contemplate is an S^1 fibration over S^1 . For example, we can study a fibration with a monodromy which is equal to a T-duality:

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

This is only allowed in bosonic string theory: in type II string theory this would exchange IIA and IIB, and so is not a symmetry of the string theory.

In particular, we can try to build this fibration via an asymmetric orbifold: if the base Y has circumference $2\pi R_y$, and the fiber is at the self-dual radius, the orbifold is realized as a \mathbb{Z}_2 shift taking $Y \rightarrow Y + \pi R_y, \mathbb{X} \rightarrow g^{-1}\mathbb{X}$.⁶

2.3 T^3 with NS-NS flux, and its T-duals

A nice example which illustrates this formalism is a T^3 with N units of NS-NS flux, discussed at length in [20, 30, 54]. We will think of this as a T^2 fibration over a base S^1 . Note that such a T^3 is not in general a solution to the equations of motion; the beta functions for this theory do not vanish. As noted in footnote 1 of [54], we can embed this in a string compactification by allowing the moduli of the T^3 to vary over additional spacetime directions.

We can describe the T^3 by three periodic coordinates x, y, z with periods 1. We will call x the base coordinate and let y, z correspond to the T^2 fibers. The T^3 can be described in the doubled torus formalism if we write

$$\begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} = \begin{pmatrix} y \\ z \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \tag{2.19}$$

as the doubled fiber coordinates.

We will consider constant flux, and choose a gauge such that B is polarized entirely in the (y, z) directions, so that $B_{yz} = Nx$. As $x \rightarrow x + 1$, B shifts by N units. The

⁶Note that the corresponding asymmetric orbifold constructed in [56] is not modular invariant, as those authors point out. However, the background is locally flat and free of fixed points; in such cases, Hellerman and Walcher argue that a modular-invariant asymmetric orbifold can always be constructed [68]. We would like to thank S. Hellerman for sharing the results of their work.

monodromy matrix is given by (2.12). If the metric G is flat and diagonal, the full metric \mathcal{H} can be written as

$$\mathcal{H} = \begin{pmatrix} 1 + (Nx)^2 & 0 & 0 & Nx \\ 0 & 1 + (Nx)^2 & -Nx & 0 \\ 0 & -Nx & 1 & 0 \\ Nx & 0 & 0 & 1 \end{pmatrix}. \quad (2.20)$$

A single T-duality acting on the y direction of this fibration transforms the monodromy matrix (2.12) to

$$g_{NT} = g_T^{-1} g_N g_T = \begin{pmatrix} 1 & -N & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & N & 1 \end{pmatrix}, \quad (2.21)$$

and the corresponding T-dual metric is

$$\mathcal{H}_T = g_T^t \mathcal{H} g_T = \begin{pmatrix} 1 & -N & 0 & 0 \\ -N & 1 + (Nx)^2 & 0 & 0 \\ 0 & 0 & 1 + (Nx)^2 & N \\ 0 & 0 & N & 1 \end{pmatrix}, \quad (2.22)$$

corresponding to a vanishing B -field and a metric of the form

$$ds^2 = dx^2 + (dy - Nxdz)^2 + dz^2. \quad (2.23)$$

This is the twisted torus background supporting “geometric” flux as described in [20, 30, 83].

In the doubled torus picture, the *components* of \mathcal{H} transform by the monodromy element (2.21), however, the *tensor* \mathcal{H} is single-valued on the $T^{2n} = T^4$ fibration. Since the example discussed in this section is geometric, the monodromy satisfies the more restrictive properties that it acts only on the components of G and that the tensor G is single-valued on the geometrical $T^n = T^2$ fibration. From this single-valuedness of G , we deduce the coordinate transition functions discussed in [30]:

$$(x, y, z) \sim (x, y + 1, z) \sim (x, y, z + 1) \sim (x + 1, y + Nz, z). \quad (2.24)$$

We can now imagine performing a second T-duality, this time along the z direction. It is not immediately clear from the above discussion that this is possible: the isometry along the z direction seems to be broken by (2.24). On the other hand, if we started with the original T^2 fibration, for which the periodicities and the metric were invariant under infinitesimal shifts in the (y, z) direction, there would seem to be nothing stopping us from performing a duality transformation

$$g_{T2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2.25)$$

If we perform a T-duality by this matrix, fiber by fiber, the monodromy matrix becomes

$$g_{NT2} = g_{T2}^{-1} g_N g_{T2} = \begin{pmatrix} 1 & 0 & 0 & -N \\ 0 & 1 & N & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.26)$$

This is a nontrivial element of $O(2, 2; \mathbb{Z})$. The metric \mathcal{H} becomes

$$\mathcal{H}_{NT2} = g_{T2}^t \mathcal{H} g_{T2} = \begin{pmatrix} 1 & 0 & 0 & -Nx \\ 0 & 1 & Nx & 0 \\ 0 & Nx & 1 + (Nx)^2 & 0 \\ -Nx & 0 & 0 & 1 + (Nx)^2 \end{pmatrix}, \quad (2.27)$$

corresponding to a NS-NS B field of the form $B_{yz} = Nx/(1 + (Nx)^2)$ and a metric of the form

$$ds^2 = dx^2 + \frac{dy^2 + dz^2}{1 + (Nx)^2}. \quad (2.28)$$

In other words, the volume of the torus shrinks as one moves around the S^1 . This is a candidate for a nongeometric compactification. While it is locally geometric, the fibration globally has a monodromy which is a nontrivial element of the T-duality group.

3. D-branes on T-folds

3.1 Geometric D-branes on T^n

Consider a worldsheet Σ with boundary $\partial\Sigma$. We will assume that the boundary conditions are those derived by varying the action (2.1), and demand that they be consistent with the self-duality constraints (2.3).

We parameterize the boundary by the worldsheet coordinate σ^0 , and the normal direction by σ^1 . The boundary terms that arise from varying the action (2.1) with respect to \mathbb{X} are

$$\delta S_{\text{bound}} = \delta \mathbb{X}^I (\mathcal{H}_{IJ} \partial_1 \mathbb{X}^J + \mathcal{J}_{IA} \partial_1 Y^A) |_{\partial\Sigma} = 0. \quad (3.1)$$

Let $\Pi_{D,J}^I$ be a projector onto Dirichlet directions, i.e., directions perpendicular to the Dirichlet branes. Then, $\Pi_D \delta \mathbb{X} = \delta \mathbb{X} \Pi_D^t = 0$. Using $\mathbf{1} = \Pi_D^t + (1 - \Pi_D^t)$, and defining $(\mathbf{1} - \Pi_D^t)_{J^I} \equiv \Pi_{N,J^I}$, we find that

$$\delta S_{\text{bound}} = \delta \mathbb{X}^I \Pi_{N,I^J} (\mathcal{H}_{JK} \partial_1 \mathbb{X}^K + \mathcal{J}_{JA} \partial_1 Y^A) = 0. \quad (3.2)$$

or

$$\Pi_{N,I^J} (\mathcal{H}_{JK} \partial_1 \mathbb{X}^K + \mathcal{J}_{JA} \partial_1 Y^A) = 0. \quad (3.3)$$

Note that the Dirichlet and Neumann directions are orthogonal with respect to \mathcal{H} . Note also that the D-brane is not completely specified by Π_D : only the directions perpendicular to the brane are. The actual position of the brane is undetermined at this level.

These boundary conditions are consistent with the self-duality constraint (2.3) if

$$\Pi_{N,I^J} L_{JK} \partial_0 \mathbb{X}^K = 0, \quad (3.4)$$

which is guaranteed if

$$\Pi_{N,I}{}^J L_{JK} = L_{IJ} \Pi_{D,K}^J. \quad (3.5)$$

Eq. (3.5) implies (as asserted in [54]) that the D-branes are null with respect to L : Using the fact that

$$\Pi_D^t \Pi_N = \Pi_D^t (\mathbf{1} - \Pi_D^t) = 0, \quad (3.6)$$

we can multiply (3.5) on the left by Π_D^t or on the right by Π_N^t to find the conditions

$$\Pi_D^t L \Pi_D = 0, \Pi_N L \Pi_N^t = 0. \quad (3.7)$$

These two conditions are not identical (as we will see in an example) and both must be imposed. Additionally, the boundary conditions imply that the classical stress tensor is conserved at the boundary:

$$T_{01}|_{\partial\Sigma} = 0. \quad (3.8)$$

This remains true for the self-dual subset of solutions to the equations of motion.

Eq. (3.4) indicates that for each Neumann condition, there is a Dirichlet condition related by an action of L . Therefore, there are always n Neumann directions and n Dirichlet directions on the doubled torus T^{2n} —the rank of Π_N and Π_D must each be n —and these directions each form a null T^n subspace of T^{2n} , in the sense of (3.7). Globally, Π_N and Π_D each define a projection from the T^{2n} bundle to a null T^n bundle over N . However, there is an important distinction between the Neumann and Dirichlet T^n bundles. Only the Neumann T^n bundle, which is wrapped by the brane, need exist as a subbundle of the T^{2n} bundle.

In summary: A D-brane in the doubled torus formalism wraps a null $T^n \subset T^{2n}$ subbundle over a cycle S of the base N .

Note that this entire discussion is at lowest order in α' and g_s , which is the order at which we understand the doubled torus formalism. There is no guarantee these branes will be stable once higher-order corrections are included.

3.1.1 T-duality acting on D-branes

We have seen that D-branes are specified by the Dirichlet projector $\Pi_{D,J}^I$. The index structure determines the transformation properties of this brane under an $O(n, n; \mathbb{Z})$ transformation g :

$$\Pi_D \rightarrow \Pi'_D = g^{-1} \Pi_D g. \quad (3.9)$$

3.1.2 Example: boundary states in the $c = 1$ Gaussian model and “nongeometric” branes

The $c = 1$ Gaussian model corresponding to a target space circle at radius R is described in our formalism as a T^2 with metric

$$\mathcal{H} = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix}. \quad (3.10)$$

The null directions are specified by a matrix Π_D which: (i) is a projector, $(\Pi_D)^2 = \Pi_D$; and (ii) satisfies (3.7). A simple calculation reveals that the solutions to the first equation in (3.7) are:

$$\begin{aligned}\Pi_D^{(1)}(a) &= \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \\ \Pi_D^{(2)}(b) &= \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}.\end{aligned}\tag{3.11}$$

These do not specify physical boundary conditions after (2.3) is solved for X . For example, $\Pi^{(1)}(a)$ leads to the boundary conditions

$$\delta X + a\delta\tilde{X} = 0, \quad -aR^2\partial_1 X + \frac{1}{R^2}\partial_1\tilde{X} = 0 \Rightarrow -aR^{-2}\partial_1 X + \partial_0 X = 0.\tag{3.12}$$

However, from the second equation in (3.7), we find that only $\Pi_D^{(1)}(0), \Pi_D^{(2)}(0)$ are allowed. These correspond to fully Dirichlet or fully Neumann conditions on the circle. Note that Π_D does not fully specify the moduli of these D-branes; the D0-brane can be at any point on the circle, and the Wilson line on the D1-brane can have any value. T-duality in this case is described by the matrix

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\tag{3.13}$$

It is easy to see that this transforms Dirichlet into Neuman conditions, and vice-versa.

Friedan [69] has characterized the space of conformal boundary conditions for $c = 1$ conformal field theories. These boundary states have been constructed in [70–73]. They consist of the Dirichlet and Neumann conditions, together with a continuous family of additional boundary states, which have finite boundary entropy (and thus finite tension) only when the radius of the circle is a rational number times the self-dual radius [73]. These additional boundary states appear to be deformations of single and multiple D-branes by boundary potentials. By moving in the family of boundary states we may interpolate between Dirichlet and Neumann conditions. It would be interesting to describe these states in the doubled torus formalism; in the open string channel, they would require including explicit boundary terms in the Lagrangian of the theory.

3.2 Torus fibrations

We would now like to study D-branes on T^n fibrations over an S^1 base, inspired by the example of a T^3 with NS-NS 3-form flux. In this case there are restrictions on the allowed D-branes, due to the nontrivial monodromy. As we have seen above, geometric D-branes on a doubled torus T^{2n} correspond to null n -planes. If these are to have finite energy, they must wrap an n -cycle of the T^n . These cycles are topologically distinct and are labeled by integers.

If the D-brane sits at a point on the base S^1 there are no restrictions on this n -plane, other than that it corresponds to a physical, conformally-invariant boundary condition. As noted by Hull [54], if we transport this brane a full period about the S^1 , the metric has changed but the topological type of the D-brane has not (the open string moduli — the

position of the D-brane, or the value of a Wilson line modulus — may, of course, change). A monodromy transformation will act on \mathcal{H}, \mathcal{J} as in (2.5). It will also act on Π_D as in (3.9). Therefore, Dp -branes can transmute into Dp' -branes as they go once around the base, or their dimensions may remain constant, but the topological class of the T^n which they wrap will not change. For such D-branes, the classical moduli space⁷ of positions on the base is not the S^1 but rather an m -fold cover,⁸ where m is the smallest integer such that $g^m \Pi_D g^{-m} = \tilde{\Pi}_D$ imposes the same boundary conditions as Π_D . (As we will see in an example below, this does not require that $\Pi_D = \tilde{\Pi}_D$. Rather, it merely requires that the row vectors of the matrix $\tilde{\Pi}_D$ be linear combinations of the row vectors of Π_D .) It is possible that m is infinity. When one includes the moduli from Wilson lines and transverse positions on the physical T^n fiber, the classical moduli space is conveniently described in a polarization-independent manner as the Dirichlet T^n fibration over the cycle mS^1 ; likewise, for D-branes located at points on the base N in a more general T-fold, the space of transverse positions on the base is mN for some m , and the classical D-brane moduli space is the Dirichlet T^n fibration over mN .⁹

If the D-brane wraps the base, then a similar argument restricts the allowed D-branes: The D-brane must close on itself. This cannot happen if, after one follows the brane around the circle, the boundary conditions are not invariant under the monodromy transformation. However, it is possible that the D-brane is wrapped m times around the base, e.g. it only closes in on itself after one goes around the circle m times. In this case, we need simply demand that there exists an m where

$$g^{-m} \Pi_D g^m = \tilde{\Pi}_D. \tag{3.14}$$

has the same space of zero eigenvectors as the original projector. Note that for higher-dimensional base manifolds N this condition holds for any cycle γ of N , topologically trivial or nontrivial, which lies inside a D-brane.

For a D-brane that wraps the entire base manifold m times, the worldvolume in the doubled geometry is Neumann T^n bundle over mN , i.e., the pullback of Neumann T^n bundle under the map $mN \rightarrow N$. The classical moduli space is the space of Wilson lines on this bundle. In the case that the base is S^1 , it is tempting to try to give this space a

⁷By “classical moduli space” we mean the space of classically allowed boundary conditions. At higher order in α' or g_s , the desired probes may break supersymmetry, be unstable, or develop a potential along the S^1 , meaning that there is strictly speaking no moduli space. Furthermore, in the example with H -flux that we discuss in section 3.4, the full string theory will have varying dilaton and the probe will experience a potential in the directions transverse to the torus; the term “moduli space” is then used loosely here to refer to approximately flat directions in the full α' and g_s corrected theory. In the weak-coupling region of such configurations, the space of classical boundary conditions should be one good measure of the geometry seen by the D-brane probe.

⁸Here we are ignoring the \mathbb{Z}_2 issue of orientation (cf. section 3.5). In some cases orientation considerations require a moduli space that is a $2m$ - rather than m -fold cover of S^1 .

⁹This statement holds only for the classical moduli space, in the sense of footnote 7. Since the true moduli space of Wilson lines is the space of *flat* connections, the Wilson lines are lifted for any S^1 factors of the T^n fiber that get a nontrivial S^1 connection over N . For example, in the background (2.28), we cannot turn on a Wilson line in the y -direction without energy cost, since even for constant A , the field strength $F = d(A_y(dy - Nxdz))$ is nonzero. This lifting is a 1-loop effect on the worldsheet.

more explicit description along the lines of “the Dirichlet T^n bundle over the circle dual to mS^1 .” However, within the T-fold formalism, where the fiber alone is doubled, the latter is not defined. Rather, the formalism of ref. [50], where both the base and fiber are doubled, is needed to make the statement more precise.

There is one subtlety with condition (3.14): the boundary conditions describing the coordinates of the brane may be invariant under the monodromy transformation, while the orientation of the submanifold may not be. In the case that the fibration is geometric, the corresponding D-brane would be wrapping an unoriented surface. This is an issue for type II D-branes: such D-branes will only be allowed in the presence of an appropriate orientifold. Other than a couple of comments in section 3.5, we will leave such additional consistency conditions for future work.

3.3 The nonperturbative topology of T-folds

At weak string coupling, D-branes are known to be excellent nonperturbative probes of distances below the string scale [74–76]. Therefore, they would seem to be an excellent probe of “stringy” nongeometric compactifications. “D-geometry”, as opposed to “stringy” geometry, typically refers to the moduli space of the D-brane probe. It could also refer to the configuration space of the worldvolume theory of the probe, as accessed by finite-energy scattering [76].

What we find¹⁰ is that there is also a “D-topology” which is distinct from the “stringy” topology of the sigma model target space. In particular, the stringy topology for T^n fibrations over S^1 is completely defined by the $O(n, n; \mathbb{Z})$ monodromy, and is nongeometric if g is not generated by $GL(n; \mathbb{Z})$ transformations and B -field shifts. However, the classical moduli space of D0-branes will always be described by fibrations with geometric monodromies.

To see this, imagine a D0-brane on a T^n fibration of S^1 with a nongeometric monodromy g . Transport the D0-brane around the S^1 base; it will return to its image as defined by $\tilde{\Pi}_D = g^{-1}\Pi_D g$. Such a monodromy will transform a D0-brane into some other brane. The moduli space is at best a multiple cover of the fibration. If there is a smallest m such that $g^{-m}\Pi_D g^m$ has the same invariant subspace as Π_D , then the D0-brane will return to itself after being transported around the circle m times. However, in this case g^m cannot be a T-duality monodromy, or it would not leave the D0-brane invariant. It must therefore be a geometric monodromy, a combination of $GL(n; \mathbb{Z})$ transformations of the T^n and B -field shifts. If the S^1 has radius R , the moduli space of the D0-brane will be a geometric T^n fibration — the Dirichlet T^n bundle — over a circle of radius mR .

3.4 Example: T^3 with H -flux and its T-duals

For our first example we will return to the T^3 with flux studied in section 2.3. The monodromy matrix is given in (2.12), and we can use this to check whether the invariant subspace of particular Dirichlet projectors survives the monodromy transformation.

¹⁰As do the authors of [76] in studying finite-energy D0-brane scattering in ALE spaces.

First, pick both Dirichlet directions in the “physical” X^i directions, corresponding to

$$\Pi_{D,1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{3.15}$$

where $\mathbf{1}$ is the 2×2 identity matrix and $\mathbf{0}$ is a 2×2 block of zeroes. Π_D specifies a brane sitting at a point in the fiber; if the brane wraps the base, it is a $D1$ -brane. Upon encircling the base, $\Pi_{D,1}$ transforms as

$$\tilde{\Pi}_{D,1} = g^{-1}\Pi_{D,1}g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & N & 0 & 0 \\ -N & 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Since the third and fourth rows of $\tilde{\Pi}_D$ are simply multiples of the second and first rows respectively, the two projectors have the same invariant subspace and define equivalent D-branes. The original boundary conditions are $\Pi_{D,1}\partial_0\mathbb{X}^I = \partial_0X^i = 0$, and the transformed boundary conditions are

$$\tilde{\Pi}_{D,1}\partial_0\mathbb{X}^I = \begin{pmatrix} \partial_0y \\ \partial_0z \\ N\partial_0z \\ -N\partial_0y \end{pmatrix} = 0, \tag{3.17}$$

which are equivalent. Thus we see that a $D1$ -brane wrapping the base is allowed.

Next, consider a $D2$ -brane wrapping the base and the y direction in the fiber, and take the Dirichlet projector to be along the y and \tilde{z} directions,

$$\Pi_{D,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.18}$$

It is easy to check that in this case, $\tilde{\Pi}_{D,2} = \Pi_{D,2}$. This $D2$ -brane is allowed, since the boundary conditions are trivially the same. Similarly, the projection onto the z and \tilde{y} directions gives an allowed $D2$ -brane. One can also check, however, that the projection onto the y, \tilde{y} or z, \tilde{z} directions does not give invariant boundary conditions.

Finally, consider the case of a $D3$ -brane wrapping both directions of the fiber. The projector here is

$$\Pi_{D,3} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{3.19}$$

which transforms to

$$\tilde{\Pi}_{D,3} = g^{-1}\Pi_{D,3}g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & N & 1 & 0 \\ -N & 0 & 0 & 1 \end{pmatrix}. \tag{3.20}$$

The original boundary conditions were $\Pi_{D,3}\partial_0\mathbb{X}^I = \partial_0\tilde{X}^i = 0$, but the transformed boundary conditions are $-N\partial_0z + \partial_0\tilde{y} = N\partial_0y + \partial_0\tilde{z} = 0$. These are no longer satisfied, so we see that a D3-brane wrapping the entire T^3 is not allowed.

It is important to realize that the correct definition of the Neumann boundary conditions is **not** $\Pi_N\partial_1\mathbb{X} = 0$, but instead $\Pi_N\mathcal{H}\partial_1\mathbb{X} = 0$, as implied by varying (2.1). For example, let us study one of the above D-branes wrapping the S^1 . The Dirichlet conditions $\Pi_D\partial_0\mathbb{X} = 0$ are invariant under the monodromy, as are the conditions $\Pi_N\mathcal{H}\partial_1\mathbb{X} = 0$. Since \mathcal{H} can, and in this case does, depend on the coordinate along the base, $\Pi_N\partial_1\mathbb{X}$ will change as we circle the S^1 . For the Neumann projector

$$\Pi_{N,1} = 1 - \Pi_{D,1}^t = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{3.21}$$

with \mathcal{H} specified in (2.20), the boundary conditions $\Pi_N\partial_1\mathbb{X}$ are $Nx\partial_1z - \partial_1\tilde{y} = Nx\partial_1y + \partial_1\tilde{z} = 0$. As $x \rightarrow x + 1$, these equations will vary and manifestly not have the same solutions.

We conclude this example by investigating the backgrounds T-dual to this one, and checking that the allowed branes are what we would have naively expected from T-duality.

3.4.1 One T-duality

T-dualizing along the y -direction gives a background with monodromy (2.21). Everything is as expected, as can be seen by switching $y \leftrightarrow \tilde{y}$ in the Dirichlet projectors above. The projections onto the (y, z) , (\tilde{y}, z) , and (\tilde{y}, \tilde{z}) give consistent boundary conditions. The projection onto the (y, \tilde{z}) directions do not. Thus, we are allowed to have D-branes of all dimensions in this background.

Since this background is geometric — the twisted T^3 of eq. (2.23)—the spectrum of D-branes can be checked directly. The cohomology of the twisted T^3 is like that of T^3 except that the global 1-form in the y -direction is $e^y = dy - Nxdz$, with $de^y = -Ndx \wedge dz$. So, $dx \wedge dz$ generates a \mathbb{Z}_N torsion factor in H^2 and $e^z \notin H^1$. Consequently, the homology is like that of T^3 except that S_y^1 is a \mathbb{Z}_N torsion class, and there is no xz -cycle (no section of the S_y^1 fibration). Thus, the possible branes wrapping the S_z^1 base are $D1_x$, $D2_{xy}$ and $D3_{xyz}$, with $D2_{xz}$ not allowed.

In the doubled formalism, these results translate into the geometrical statement that there exist null subbundles $T_{\tilde{y}\tilde{z}}^2$, $T_{y\tilde{z}}^2$ and T_{yz}^2 , but not $T_{z\tilde{y}}^2$. This can be checked directly as well. The metric of the T^4 fibration over S^1 is

$$ds^2 = dx^2 + ((e^y)^2 + dz^2) + (d\tilde{y}^2 + (e^{\tilde{z}})^2), \tag{3.22}$$

where $e^{\tilde{z}} = d\tilde{z} + Nxd\tilde{y}$. The quantity in the first set of parentheses is the physical T^2 fiber metric, and that in the second set of parentheses is the dual (inverse) T^2 metric. Thus, H^2 of the doubled geometry contains $e^y \wedge dz$, $dz \wedge d\tilde{y}$ and $d\tilde{y} \wedge de^{\tilde{z}}$, but not $e^y \wedge e^{\tilde{z}}$, since latter is not closed. The homology dual of this statement exactly reproduces the correct list of null T^2 subbundles.

This example illustrates the fact that the Dirichlet T^n bundle need not be a subbundle of the T^{2n} bundle. Consider the D-brane that wraps the $T_{y\tilde{z}}^2$ subbundle. The metric (3.22)

projects to the product metric on $S_x^1 \times T_{y\tilde{z}}^2$ for the Neumann bundle, and to the product metric on $S_x^1 \times T_{z\tilde{y}}^2$ for the Dirichlet bundle. However, only the Neumann bundle is a subbundle of the doubled geometry.

3.4.2 Two T-dualities

Now consider the background with monodromy (2.26), which we get by additionally T-dualizing in the z -direction. It is straightforward to check that the allowed projections are onto the (y, \tilde{z}) , (\tilde{y}, z) , and (\tilde{y}, \tilde{z}) directions, corresponding to two D2-branes and one D3-brane. Since there were no D3-branes in the original H -flux background, there are no D1-branes here.¹¹

The moduli space of D0-brane positions on the S^1 base in this background is an infinite cover of the S^1 . To see this, consider the image of the D0-brane under T-duality, in the original background with H -flux. This is a D2-brane wrapping the T^2 fiber. After transport around the S^1 base, the B -field will induce N units of D0-brane charge.

3.5 Example: asymmetric orbifolds of T^3

As a final example, let us consider an orbifold of T^3 which combines a shift on the base with an action on the moduli of the T^2 fiber. To begin, consider orbifolding by the action

$$\begin{cases} \tau \rightarrow -1/\tau, \\ x \rightarrow x + 1, \end{cases} \quad (3.23)$$

which implements a 90° rotation on the fiber upon encircling the base. This is a perfectly good geometrical (and symmetric) orbifold, but will help us with related nongeometric orbifolds momentarily. It is clear what the allowed D-branes wrapping the base are: Since this action rotates the two 1-cycles of the fiber, one expects that a D1-brane wrapping the base or a D3-brane wrapping the entire fiber will be invariant, but not a D2-brane wrapping the base once while wrapping a one 1-cycle of the fiber. We can easily see this from the projectors $\Pi_{D,N}$. The monodromy is simply

$$g_\tau = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix}, \quad (3.24)$$

where \mathbf{S} is a 2×2 antisymmetric matrix with 1 in the upper right corner. It is now an easy matter to show that the projectors onto both the (y, z) and (\tilde{y}, \tilde{z}) directions are invariant under this monodromy, but the projectors onto the (y, \tilde{z}) and (\tilde{y}, z) directions are not. Thus, D1-branes and D3-branes wrapping the base are allowed, but D2-branes wrapping the base once are not.

Since the orbifold (3.23) is a geometrical background, we can check this result directly. The 3-dimensional geometry is \mathbb{R}^3/Γ , where Γ is generated by the group elements α , β and γ , taking (x, y, z) to $(x + 1, z, -y)$, $(x, y + 1, z)$ and $(x, y, z + 1)$, respectively. To determine

¹¹Since it is not a T-fold, the background obtained from the T^3 with H -flux after three T-dualities is not discussed in this paper. However, note that the absence of D3 branes in the original background implies that D0-branes do not exist after three T-dualities. Therefore, such backgrounds are not even locally geometric: there are not even points on which to place D0-branes.

which cycles can be wrapped, we need the homology of \mathbb{R}^3/Γ . This is easily computed¹² to be $H_0 = H_2 = H_3 = \mathbb{Z}$ and $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$. The corresponding \mathbb{Z} -valued classes are a point, the T^2 fiber, the whole manifold, and the S^1 base. The \mathbb{Z}_2 torsion cycle is the class of the z circle fiber, which via the identification is the same as a circle fiber oriented in the $-z$ or $\pm y$ directions.

Qualitatively, the geometry can be thought of as an oriented, higher dimensional, \mathbb{Z}_4 analog of a Klein bottle. A Klein bottle is an S^1 fibration over S^1 with a \mathbb{Z}_2 orientation reversal twisting the fiber. Here, we instead have a T^2 fibration over S^1 with a \mathbb{Z}_4 rotation twisting the fiber. In fact, there exists a Klein bottle within this geometry. Since $\mathbf{S}^2 = -1$, we have $g_\tau^{-2} \Pi_D g_\tau^2 = \Pi_D$, so it appears that by eq. (3.14) a D2 brane can twice wrap the base *twice* while wrapping a 1-cycle of the fiber. However, this D-brane is wrapping an unoriented cycle — more precisely, a Klein bottle: Such a configuration will be allowed in the bosonic string, and certain orientifolds of the type II string.

T-dualizing this example on one of the fiber directions interchanges the complex structure and Kähler moduli. The result is a manifestly nongeometric orbifold with action

$$\begin{cases} \rho \rightarrow -1/\rho, \\ x \rightarrow x + 1, \end{cases} \quad (3.25)$$

which mixes B -field and metric. It is also clearly an asymmetric orbifold [53, 56, 77–81]: the combined action $\tau \rightarrow -1/\tau, \rho \rightarrow -1/\rho$ is the same as two T-dualities, so the action $\rho \rightarrow -1/\rho$ is just the asymmetric T-duality action combined with a symmetric 90° rotation, and as such is still asymmetric. Since this orbifold is just one T-duality away from the $\tau \rightarrow -1/\tau$ orbifold where the only allowed branes wrapping the base once were D1-branes and D3-branes, here we only expect D2-branes to be allowed. The appropriate monodromy matrix is

$$g_\rho = \begin{pmatrix} \mathbf{0} & \mathbf{S} \\ \mathbf{S} & \mathbf{0} \end{pmatrix}. \quad (3.26)$$

One can easily check that only the Dirichlet projectors onto the (y, \tilde{z}) and (\tilde{y}, z) directions give invariant boundary conditions, confirming our intuition.

Finally, let us combine these two actions into an asymmetric orbifold that is not dual to a symmetric orbifold,¹³

$$\begin{cases} \rho \rightarrow -1/\rho, \\ \tau \rightarrow -1/\tau, \\ x \rightarrow x + 1. \end{cases} \quad (3.27)$$

Consider D-branes that wrap the base exactly once. As stated above, the action (3.27) is the same as completely T-dualizing the fiber. Thus, we expect no such D-branes to exist

¹²Clearly, $H_0 = H_3 = \mathbb{Z}$. The fundamental group π_1 is Γ , with nonzero commutators $\alpha\beta\alpha^{-1}\beta^{-1} = \gamma^{-1}\beta^{-1}$ and $\alpha\gamma\alpha^{-1}\gamma^{-1} = \beta\gamma^{-1}$. Given π_1 , the group $H_1(\mathbb{Z})$ is obtained by setting all commutators to unity. This gives $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$, from α and β , respectively. Finally, $H_2 = \mathbb{Z}$ by Poincaré duality together with the the Universal Coefficient Theorem $H_n^{\text{torsion}} = H_{n+1}^{\text{torsion}}$.

¹³In [56], the naïve presentation of this orbifold for the heterotic string was shown to not be modular invariant. Whether there exists a consistent prescription is the subject of current study [68]. We proceed with this example for illustrative purposes.

in this background. The appropriate monodromy here is

$$g_{\tau,\rho} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{3.28}$$

and one can check that there are no projectors which produce invariant boundary conditions. Alternatively, one can see that there are no null T^2 subbundles of the doubled geometry. For such a subbundle, the T^2 fiber must be preserved by the monodromy matrix. In this background, the monodromy takes $X^i \rightarrow -\tilde{X}_i$ and $\tilde{X}_i \rightarrow -X^i$. So, X_L and X_R are eigenvectors of the monodromy; for each i , the subspaces of definite handedness are monodromy invariant. But, the $SO(2, 2)$ invariant metric is $ds_L^2 = 2dX^i d\tilde{X}_i = dX_L^2 - dX_R^2$, so these subspaces are not null. Therefore, there is no null T^2 subbundle, and no D-brane wrapping the base once. On the other hand, since the monodromy squares to the identity, there do exist D-branes that wrap the base twice.¹⁴

Now consider D0-branes. It is worthwhile to consider the D-geometry of this example in two different descriptions, either as a fibration with monodromy or as an asymmetric orbifold. As a fibration, we see that a D0-brane must go around the base twice in order to come back to itself. Thus, the moduli space of a D0-branes should be a T^3 with one circle of radius $2R$ (the base) and two circles of self-dual radius R_{sd} (the fiber). From the asymmetric orbifold description, we would simply start with a base with radius $2R$, and orbifold by $\tau \rightarrow -1/\tau$ and $\rho \rightarrow -1/\rho$ as $x \rightarrow x + \pi R$. The moduli space of a D0-brane on this is the same as the original space, the T^3 with radii $2R$, R_{sd} , and R_{sd} . The D-geometry is the same in either description.

4. Conclusions

We have described simple examples of T-folds with local fiber geometry T^2 and base S^1 . These T-folds are interesting in that they generically have no associated global target space geometry, and are conveniently described using the doubled torus formalism pursued by Hull [54]. In this formalism, the background is described by a T^{2n} fibration over a base manifold N , with transition functions in $O(n, n; \mathbb{Z})$. In each patch, the physical coordinates X^i and T-dual coordinates \tilde{X}_i each span null T^n subspaces determined by a choice of polarization $GL(n; \mathbb{Z})$ in $O(n, n; \mathbb{Z})$. Similarly, a D-brane wraps a null T^n bundle — the Neumann bundle — over a cycle of the base. Using this formalism, we have determined the spectrum of D-branes compatible with each of our T-fold examples. When any such D-branes exist, we can associate true global geometries to a nongeometric background through the classical moduli spaces seen by these branes. We have discussed aspects of these D-brane moduli spaces and described them explicitly to the extent possible. The sharpest statement can be made when a D-brane is located at point on the base. In this case the classical moduli space is the Dirichlet T^n bundle over m -fold cover of the base.

Three clear avenues for further work were mentioned in the paper:

¹⁴One could try to think of these branes as once-wrapped D2-D2 or D1-D3 bound states. However, since the constituent branes do not exist in this background, these objects are not strictly speaking bound states.

1. We would like to better understand quantum dynamics in the doubled torus formalism.
2. Examples on more general base manifolds N should be studied. This is particularly important for understanding mirror symmetry with NS-NS flux.
3. As Hull points out [54], the doubled torus description of geometric twisted tori is a real analog of the description of generalized Calabi-Yau manifolds [59, 60]. The doubling of the torus corresponds to the use of $TM \oplus T^*M$ for generalized Calabi-Yau manifolds. It is only a real analog — in a sense because M is the fiber rather than the entire manifold. In ref. [50], the doubling was extended to include the base, at least in the case that the base is a (doubled) S^1 . More generally, in the case that an even-dimensional underlying structure exists that can play the role of compactification *geometry* [82], we would like to see this connection to generalized complex geometry explored further and made precise.

An additional direction to pursue is the following:

4. In this article, 3-form flux corresponded to a T^2 fibration over S^1 with a particular monodromy. Even within this framework, there exist other monodromies corresponding to “nongeometric” fluxes as studied in [20]. These are interesting in their own right, and also give new opportunities for solving tadpole constraints in flux compactifications.

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